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THE EIGENVALUE DISTRIBUTION FOR A DEGENERATE ELLIPTIC OPERATOR $-\operatorname{div}\{(1-|x|^2)\operatorname{grad}\cdot\}$

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ABSTRACT

We sharpen the remainder estimate of the asymptotic formula for the eigenvalue distribution for the degenerate operator $-\operatorname{div}\{(1-|x|^2)\operatorname{grad}\cdot\}$ in the unit ball of the Euclidean space $\mathbf{R}^n$. In particular we find the second term when $n=2$.

1. Introduction

In this paper we will study the remainder term of the asymptotic formula for the eigenvalue distribution for the degenerate elliptic operator $-\operatorname{div}\{(1-|x|^2)\operatorname{grad}\cdot\}$ in the unit ball of the Euclidean space $\mathbf{R}^n$ when $n \geqq 2$. Particularly we will find the second term when $n=2$.

The degenerate elliptic operators are considered as the generalization to the multi-dimension of Legendre's operator

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}$$

in the interval $(-1,1)$. The eigenvalue distribution of the non-degenerate elliptic operators, such as the Laplacian under the Dirichlet boundary condition, has been studied by many authors since H. Weyl's work. The spectra of the degenerate operators were studied for the first time by Baouendi and Goulaouic [2] who treated the operators of the second order. They found that the counting function of the degenerate operator grows more rapidly than that of the non-degenerate operator when $n \geqq 2$.

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Using the min–max principle, Nordin [5] obtained the principal term of the asymptotic formula of the counting function for the degenerate operator of the second order. Independently Shimakura [9] also derived the principal term for the operator $-\mathrm{div}\{(1-|x|^2)\mathrm{grad}\cdot\}$.

Lai [6],[7],[8] generalized their results to the case of the operator of order $2m$ which is elliptic in the interior, and degenerates with the degree of order m at the boundary of a bounded C^∞-domain in $\mathbf{R}^n$, and at the same time he sharpened the estimate of the remainder term. He obtained the formula for the counting function $N(t)$ as $t \to \infty$:

$$(1.1) \qquad \begin{aligned} N(t) &= c_2 t^{1/m} \log t^{1/m} + O(t^{1/m}) \qquad \text{when } n = 2, \\ N(t) &= c_n t^{(n-1)/m} + O(t^{(n-1-\theta)/m}) \\ &\qquad\qquad \text{for any } \theta \in [0, 1/6) \quad \text{when } n \geqq 3, \end{aligned}$$

where c_n is the constant.

In this paper we will improve the above result in case of the operator $-\mathrm{div}\{(1-|x|^2)\mathrm{grad}\cdot\}$. Our result is:

$$\begin{aligned} N(t) &= c_2 t \log t + c_2' t + O(t^{1/2}) \qquad \text{when } n = 2, \\ N(t) &= c_n t^{n-1} + O(t^{n-3/2}) \qquad \text{when } n \geqq 3, \end{aligned}$$

where c_2' is the constant.

To prove our result we follow Shimakura's idea. He obtained the explicit formula for the eigenvalues of the operator $-\mathrm{div}\{(1-|x|^2)\mathrm{grad}\cdot\}$, and using this formula he derived the principal term of the asymptotic formula. Our method is an elaboration of Shimakura's method. This explicit formula for the eigenvalues plays an important role in our proof.

2. Notation and result

Let Ω be the unit ball of the Euclidean space $\mathbf{R}^n$. We consider the degenerate elliptic operator A_0 in $L^2(\Omega)$ defined by

$$A_0 u = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\{\left(1-|x|^2\right)\frac{\partial u}{\partial x_j}\right\} + (n-1)u$$

with the domain of definition

$$D(A_0) = H_1^2(\Omega) = \{u \in D'(\Omega);\ (1-|x|^2)D^\alpha u \in L^2(\Omega)\ (|\alpha| \leqq 2)\}.$$

Here and in what follows we use the notations

$$D_j = -\sqrt{-1}\,\frac{\partial}{\partial x_j} \qquad (j = 1,2,\ldots,n),$$

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n} \qquad \text{for a multi-index } \alpha = (\alpha_1,\ldots,\alpha_n).$$

We have added the term $(n-1)u$ for the sake of convenience.

It is known that the spectrum of A_0 consists only of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ which accumulate only at ∞. When we enumerate eigenvalues, each eigenvalue is repeated the number of times equal to its multiplicity. Let $N_0(t)$ denote the counting function, that is, the number of eigenvalues of A_0 which do not exceed t. Our result is:

THEOREM 1. *The following asymptotic formulas for $N_0(t)$ hold.*

(i) *When $n = 2$,*

$$N(t) = \frac{1}{4}\,t\log t + \left(\frac{\log 2 + \gamma}{2} - \frac{1}{4}\right)t + O(t^{1/2}).$$

(ii) *When $n \geqq 3$,*

$$N(t) = \frac{2^{3-2n}\zeta(n-1;1/2)}{(n-1)!}\,t^{n-1} + O(t^{n-3/2}).$$

Here γ is Euler's constant, i.e.,

$$\gamma = \lim_{k\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \log k\right),$$

and $\zeta(z;a)$ is Hurwitz' zeta function, i.e.,

$$\zeta(z;a) = \sum_{k=0}^{\infty}(k+a)^{-z} \qquad \textit{for } \operatorname{Re} z > 1, \quad a > 0.$$

It is remarkable that we have found the second term when $n = 2$. The proof of Theorem 1 depends on the following explicit formula of the eigenvalues of A_0.

PROPOSITION 2 (Shimakura [9]). (i) *A_0 has eigenvalues $\{\lambda_{k,l}\}_{k,l}^{\infty}$ and the multiplicity of $\lambda_{k,l}$ is $\mu(k)$, where*

$$\lambda_{k,l} = (2l+1)(2l+2k+n-1) \qquad \textit{for } k,l = 0,1,2,\ldots,$$

$$\begin{cases} \mu(0) = 1, \mu(k) = 2 & \textit{for } k \geqq 1 \textit{ when } n = 2, \\ \mu(k) = (2k+n-2)\dfrac{(k+n-3)!}{k!\,(n-2)!} & \textit{for } k \geqq 0 \textit{ when } n \geqq 3. \end{cases}$$

(ii) *For each* (k,l) *the eigenfunction corresponding to the eigenvalue* $\lambda_{k,l}$ *is*

$$u_{k,l,j}(x) = G_l\left(k+\frac{n}{2}, k+\frac{n}{2}; |x|^2\right) H_{k,j}(x)$$

for $k,l = 0,1,2,\ldots; j = 1,2,\ldots,\mu(k)$ *where* G_l *is defined by the hypergeometric function, i.e.,* $G_l(p,q;z) = F(l+p,-l;q;z)$, *and* $\{H_{k,j}(x)\}_{j=1}^{\mu(k)}$ *is the base of the space of harmonic homogeneous polynomials of degree* k.

REMARK. Although we will not need Proposition 2 (ii) in the proof of Theorem 1, we have written it since it seems that there are some mistakes in [9].

Using Proposition 2, Shimakura derived the asymptotic formula for $N_0(t)$ with the remainder estimate $o(t\log t)$ or $o(t^{n-1})$ when $n = 2$ or $n \geqq 3$ respectively. Elaborating Shimakura's idea, we will prove Theorem 1 in the next section.

To compare our result with the known one we shall mention Lai's result briefly. Lai considers the degenerate elliptic operator in $L^2(\Omega)$ associated with a symmetric integro-differential form

$$a[u,v] = \int_\Omega \varphi(x)^m \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x) D^\alpha u(x) \overline{D^\beta v(x)}\, dx$$

where Ω is a domain such that there exists $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ which satisfies

$$\Omega = \{x \in \mathbf{R}^n; \varphi(x) > 0\},$$

$$\partial\Omega = \{x \in \mathbf{R}^n; \varphi(x) = 0\},$$

$$d\varphi(x) \neq 0 \qquad \text{on } \partial\Omega.$$

He obtained the asymptotic formula (1.1) for the counting function $N(t)$. The operator A_0 is the special case of the degenerate elliptic operators which Lai considered. Our result suggests that Lai's result can be improved. It is interesting to examine whether (1.1) can be improved as follows or not.

(i) When $n = 2$, there exists a constant c' such that

$$N(t) = c_n t^{1/m} \log t^{1/m} + c' t^{1/m} + O(t^{1/2m}).$$

(ii) When $n \geqq 3$,

$$N(t) = c_n t^{(n-1)/m} + O(t^{(n-3/2)/m}).$$

That is, we can take $\theta = \frac{1}{2}$ in (1.1).
This problem is still open.

3. Proof of Theorem 1

LEMMA 3.1. *The following asymptotic formulas hold.*
(i)

$$\sum_{j=1}^{N} \frac{1}{j} = \log N + \gamma + O(N^{-1}) \qquad \text{as } N \to \infty.$$

(ii) *For $a > 0$ and $n \geqq 3$,*

$$\sum_{j=0}^{N} (j+a)^{-n+1} = \zeta(n-1; a) + O(N^{-n+2}) \qquad \text{as } N \to \infty.$$

PROOF. Lemma 3.1 can be easily verified by using the Euler–Maclaurin summation formula (see Apostol [1] Theorem 3.2). q.e.d.

First we shall prove Theorem 1 when $n = 2$. We denote by $\nu(k,t)$ the number of l for which $\lambda_{k,l} \leqq t$ holds. Proposition 2 and simple calculation give

$$\begin{cases} \nu(k,t) = [f(k)] & \text{when } 0 \leqq k \leqq \left[\dfrac{t-1}{2}\right], \\ \nu(k,t) = 0 & \text{when } k > \left[\dfrac{t-1}{2}\right], \end{cases}$$

where

$$f(k) = \frac{-k + 1 + (k^2 + t)^{1/2}}{2}.$$

Here and in what follows, for a real number x we denote by $[x]$ the largest integer which is not larger than x. Putting $T = (t-1)/2$ and taking the multiplicity of $\lambda_{k,l}$ into account, we have

$$N(t) = \nu(0,t) + 2\sum_{k=1}^{[T]} \nu(k,t) = 2\sum_{k=1}^{[T]} [f(k)] + O(t^{1/2}).$$

We note that $\Sigma_{k=1}^{[T]}[f(k)]$ is the number of lattice points (i.e., points whose components are all integers) which are contained in the region

$$\{(x,y); 0<y\leqq f(x),\ x>0\} = \{(x,y);\ 0<x\leqq f^{-1}(y), y>0\}.$$

Hence we have

$$\sum_{k=1}^{[T]}[f(k)] = \sum_{j=1}^{[\tau]}[f^{-1}(j)]$$

where $\tau = f(1) = \{(t+1)^{1/2}\}/2$ and $f^{-1}(j)$ is the inverse function of f:

$$f^{-1}(j) = \frac{t}{2(2j-1)} - j + \frac{1}{2}.$$

Then we have

$$N(t) = t\sum_{j=1}^{[\tau]}\frac{1}{2j-1} - 2\sum_{j=1}^{[\tau]} j + O(t^{1/2}). \tag{3.1}$$

Noting

$$[\tau] = \frac{1}{2}t^{1/2}(1+O(t^{-1/2})) \qquad \text{as } t\to\infty,$$

and using Lemma 3.1(i), we get

$$\begin{aligned}\sum_{j=1}^{[\tau]}\frac{1}{2j-1} &= \sum_{j=1}^{2[\tau]}\frac{1}{j} - \sum_{j=1}^{[\tau]}\frac{1}{2j} \\ &= \log 2[\tau] + \gamma - \frac{1}{2}(\log[\tau]+\gamma) + O([\tau]^{-1}) \\ &= \frac{1}{4}\log t + \frac{\gamma+\log 2}{2} + O(t^{-1/2}),\end{aligned} \tag{3.2}$$

and

$$2\sum_{j=1}^{[\tau]} j = [\tau]([\tau]+1) = \frac{1}{4}t + O(t^{1/2}). \tag{3.3}$$

From (3.1)–(3.3) we obtain (i) of Theorem 1.

Next we shall show Theorem 1 when $n\geqq 3$. The proof is a little complicated, though it goes similarly as in the case of $n=2$. Defining $\nu(k,t)$ as before, we have

$$\begin{cases}\nu(k,t) = [f(k)] & \text{when } 0\leqq k\leqq [T], \\ \nu(k,t) = 0 & \text{when } k>[T],\end{cases}$$

where

$$T = \frac{t-n+1}{2},$$

and

$$f(k) = \frac{1}{4}\{-2k - n + 4 + (4k^2 + 4(n-2)k + n^2 - 4n + 4t + 4)^{1/2}\}.$$

We prepare several estimates which will be used later.

LEMMA 3.2. (i) *The inverse function of f is*

$$f^{-1}(l) = \frac{t}{2(2l-1)} - l - \frac{n-3}{2}.$$

(ii) *There exists a polynomial $P_{n-3}(k)$ in k of degree $n-3$ such that*

$$\mu(k) = \frac{2}{(n-2)!}k^{n-2} + P_{n-3}(k).$$

(iii) *It follows that*

$$0 \leqq f(k) \leqq t^{1/2} \qquad \text{when } 1 \leqq k \leqq [T], \quad t \geqq 1.$$

(iv) *There exists $C > 0$ such that when $t \geqq 1$,*

$$0 \leqq \sum_{k=1}^{[T]} k^i [f(k)] \leqq Ct^{n-3/2} \qquad \text{for } 0 \leqq i \leqq n-3,$$

and

$$0 \leqq \sum_{k=1}^{[T]} k^i [f(k)]^2 \leqq Ct^{n-2} \qquad \text{for } 0 \leqq i \leqq n-4.$$

(v) *There exists $C > 0$ such that when $t \geqq 1$,*

$$0 \leqq \sum_{k=1}^{[T]} k^i \sum_{l=1}^{[f(k)]} [f^{-1}(l)] \leqq Ct^{n-3/2} \qquad \text{for } 0 \leqq i \leqq n-4.$$

(vi) *There exists $C > 0$ such that when $t \geqq 1$,*

$$0 \leqq \sum_{l=1}^{[f(1)]} [f^{-1}(l)]^i \leqq Ct^{n-3/2} \qquad \text{for } 0 \leqq i \leqq n-2.$$

(vii) *It follows that as $t \to \infty$,*

$$\sum_{l=1}^{[f(1)]} [f^{-1}(l)]^{n-1} = \sum_{l=1}^{[f(1)]} \left(\frac{t}{2(2l-1)}\right)^{n-1} + O(t^{n-3/2}).$$

PROOF. (i) and (ii) are easily seen.

Let us put $K = 2k + n - 2$ (> 0). Then we have

$$0 \leqq f(k) = \frac{1}{4}\left(-K + 2 + \sqrt{K^2 + 4t}\right)$$
$$= \frac{1}{4}\left(2 + \frac{4t}{K + \sqrt{K^2 + 4t}}\right)$$
$$\leqq \frac{1}{4}\left(2 + 2t^{1/2}\right) \leqq t^{1/2},$$

from which (iii) follows.

(iv) follows easily from (iii).

In the sequel we denote by C positive constants depending only on n, which may differ from each other.

We note that $f(1) = O(t^{1/2})$ and $f^{-1}(1) = O(t)$. Since f is a decreasing function, we have

$$0 \leqq \sum_{k=1}^{[T]} k^i \sum_{l=1}^{[f(k)]} [f^{-1}(l)] \leqq \sum_{k=1}^{[T]} k^i f(1) f^{-1}(1)$$
$$\leqq C t^{i+1} t^{1/2} t \leqq C t^{n-3/2},$$

from which (v) follows.

From (i) we have

$$0 \leqq \sum_{l=1}^{[f(1)]} [f^{-1}(l)]^i \leqq \sum_{l=1}^{[f(1)]} f^{-1}(l)^{n-2}$$
$$\leqq C \sum_{l=1}^{[f(1)]} \sum_{\alpha+\beta \leqq n-2} \left(\frac{t}{2l-1}\right)^{\alpha} l^{\beta}.$$

It follows that when $\alpha > \beta$,

$$0 \leqq \sum_{l=1}^{[f(1)]} \left(\frac{t}{2l-1}\right)^{\alpha} l^{\beta} \leqq C t^{\alpha} \log(f(1) + 1)$$
$$\leqq C t^{n-2} \log(t + 1) \leqq C t^{n-3/2},$$

and when $\alpha \leqq \beta$,

$$0 \leqq \sum_{l=1}^{[f(1)]} \left(\frac{t}{2l-1}\right)^{\alpha} l^{\beta} \leqq C t^{\alpha} f(1)^{\beta-\alpha+1} \leqq C t^{\alpha + (\beta-\alpha+1)/2}$$
$$\leqq C t^{(\alpha+\beta+1)/2} \leqq C t^{(n-1)/2} \leqq C t^{n-3/2}.$$

Hence we get (vi).

To prove (vii) we may assume $[T] \geqq 1$. Noting

$$0 \leqq f^{-1}(l) - 1 \leqq [f^{-1}(l)] \leqq f^{-1}(l),$$

we have

$$\left| [f^{-1}(l)]^{n-1} - \left(\frac{t}{2(2l-1)}\right)^{n-1} \right| \leqq C \sum_{\alpha+\beta \leqq n-1, \alpha \leqq n-2} \left(\frac{t}{2l-1}\right)^{\alpha} l^{\beta}.$$

Then we obtain (vii) in the same way as in (vi). q.e.d.

Taking the multiplicity of $\lambda_{k,l}$ into account and using Lemma 3.2(ii),(iv), we have

$$N(t) = \sum_{k=0}^{[T]} \mu(k)\nu(k,t) = \sum_{k=1}^{[T]} \frac{2}{(n-2)!} k^{n-2}[f(k)] + O(t^{n-3/2}). \tag{3.4}$$

Let us evaluate $\Sigma_{k=1}^{[T]} k^{n-2}[f(k)]$. Let $Q_{n-4}(j)$ denote the polynomial in j which satisfies

$$j^{n-2} - (j-1)^{n-2} = (n-2)j^{n-3} + Q_{n-4}(j).$$

We note that $Q_{n-4}(j) \equiv 0$ when $n = 3$, and that $Q_{n-4}(j)$ is a polynomial of degree $n-4$ when $n \geqq 4$.

Simple calculations and Lemma 3.2 (iv),(v) show

$$\begin{aligned}
\sum_{k=1}^{[T]} k^{n-2}[f(k)] &= \sum_{k=1}^{[T]} \sum_{j=1}^{k} (j^{n-2} - (j-1)^{n-2})\,[f(k)] \\
&= \sum_{j=1}^{[T]} \sum_{k=j}^{[T]} (j^{n-2} - (j-1)^{n-2})\,[f(k)] \\
&= \sum_{j=1}^{[T]} \{(n-2)j^{n-3} + Q_{n-4}(j)\} \sum_{k=1}^{[T-j+1]} [f(k+j-1)] \\
&= \sum_{j=1}^{[T]} \{(n-2)j^{n-3} + Q_{n-4}(j)\} \sum_{l=1}^{[f(j)]} [f^{-1}(l) - j + 1] \\
&= (n-2) \sum_{j=1}^{[T]} \sum_{l=1}^{[f(j)]} j^{n-3}[f^{-1}(l)] \\
&\quad - (n-2) \sum_{j=1}^{[T]} j^{n-2}[f(j)] + O(t^{n-3/2}).
\end{aligned}$$

Solving the above equation for $\Sigma_{k=1}^{[T]} k^{n-2}[f(k)]$, we obtain

$$\sum_{k=1}^{[T]} k^{n-2}[f(k)] = \frac{n-2}{n-1} \sum_{j=1}^{[T]} \sum_{l=1}^{[f(j)]} j^{n-3}[f^{-1}(l)] + O(t^{n-3/2}). \tag{3.5}$$

Combining (3.4) and (3.5), changing the order of summation, and using the formula

$$(n-2) \sum_{j=1}^{J} j^{n-3} = J^{n-2} + R_{n-3}(J)$$

($R_{n-3}(J)$ is a polynomial in J of degree $n-3$),

we get

$$\begin{aligned} N(t) &= \frac{2(n-2)}{(n-1)!} \sum_{j=1}^{[T]} \sum_{l=1}^{[f(j)]} j^{n-3}[f^{-1}(l)] + O(t^{n-3/2}) \\ &= \frac{2(n-2)}{(n-1)!} \sum_{l=1}^{[f(1)]} \sum_{j=1}^{[f^{-1}(l)]} j^{n-3}[f^{-1}(l)] + O(t^{n-3/2}) \\ &= \frac{2}{(n-1)!} \sum_{l=1}^{[f(1)]} \{[f^{-1}(l)]^{n-2} + R_{n-3}([f^{-1}(l)])\}[f^{-1}(l)] + O(t^{n-3/2}). \end{aligned}$$

Finally, from Lemma 3.2 (vi),(vii) and Lemma 3.1 it follows that

$$\begin{aligned} N(t) &= \frac{2}{(n-1)!} \sum_{l=1}^{[f(1)]} \left(\frac{t}{2(2l-1)} \right)^{n-1} + O(t^{n-3/2}) \\ &= \frac{2^{3-2n}\zeta(n-1;1/2)}{(n-1)!} t^{n-1} + O(t^{n-3/2}). \end{aligned}$$

This completes the proof of Theorem 1.

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